Proposition

Suppose that a certain quantitative random variable belonging to a Normal distribution has a mean of m and a standard deviation of σ. Further, assume that after log-transforming this variable, the resulting distribution has a mean and standard deviation of $\widehat{m}$ and $\widehat{\sigma}$, respectively.

Then, there exist the following relations between these pairs,

$$m = \ln\left(\frac{\widehat{m}}{\sqrt{\frac{\widehat{\sigma}^2}{\widehat{m}^2}+1}}\right) \tag{1}$$

$$\sigma^2 = \ln\left(\frac{\widehat{\sigma}^2}{\widehat{m}^2}+1\right). \tag{2}$$

Proof

Let Y be a normal random variable with mean m and variance σ^2. In other words, by standard notation $Y \sim \mathcal{N}(m, \sigma^2)$. The probability density function (pdf) of Y is then,

$$g(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-m)^2}{2\sigma^2}\right). \tag{3}$$

Provided that a random variable X is related to Y with the following equation,

$$X = \exp(Y). \tag{4}$$

then it said to have a log-normal distribution.

We denote the expected value and variance of a log-normally distributed random variable X by $\mathbb{E}[X]$ and $\mathbb{V}[X]$, respectively. According to the preposition, $\mathbb{E}[X] = \widehat{m}$ and $\mathbb{V}[X] = \widehat{\sigma}^2$.

In order to solve for $\mathbb{E}[X]$ and $\mathbb{V}[X]$ in terms of m and σ, we make use of the following facts,

- $\mathbb{E}[X] = \int_{\mathbb{R}} x f(x) dx$
- $\mathbb{V}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

where $\mathbb{R}$ is the set of real numbers, $f(x)$ is the pdf of random variable X.

Deriving pdf of X

We know that Y admits a pdf as shown in Eq. 3. If Y and X are related with Eq. 4, then

$$f(x) = g(y)\frac{dy}{dx}. \tag{5}$$

Using Eq. 4, the derivative terms can be related with the following

$$dy = \frac{1}{x} dx. \tag{6}$$

Replacing y and dy in Eq. 5 with proper terms using Eq. 6, we obtain

$$f(x) = g\left(\ln(x)\right)\frac{1}{x},$$

which is equivalent to

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\ln(x)-m)^2}{2\sigma^2}\right)\frac{1}{x}. \tag{7}$$

Deriving expected value of X

To find $\mathbb{E}[X]$, we need to evaluate the following definite integral,

$$\mathbb{E}[X] = \int_{\mathbb{R}} x f(x) dx,$$

by replacing $f(x)$ with Eq. 7.

First of all, although the limits of the integral are $\pm\infty$ for the general case, for log-normal random variables, they are shrunk down to 0 and ∞ due to the non-negativity of the random variable. Changing the limits accordingly and replacing $f(x)$ with Eq. 7, we obtain

$$\mathbb{E}[X] = \int_0^\infty x \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\ln(x) - m)^2}{2\sigma^2}\right) \frac{1}{x} dx. \tag{8}$$

We apply a change of variables such that

$$t = \frac{\ln(x) - m}{\sigma}. \tag{9}$$

Here, the derivative terms have the following relation

$$dt = \frac{1}{\sigma x} dx. \tag{10}$$

By using Eq. 9, we can see that

$$x = \exp(\sigma t + m). \tag{11}$$

Replacing Eq. 11 in Eq. 10, we obtain

$$\sigma \exp(\sigma t + m) dt = dx.$$

Moreover, from Eq. 9, we know that $x = 0$ corresponds to $t = -\infty$ and $x = \infty$ corresponds to $t = \infty$. Arranging the limits according to this change of variables, Eq. 8 becomes

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}t^2\right) \sigma \exp(\sigma t + m) dt.$$

This expression is equivalent to,

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(t^2 - 2\sigma t + \sigma^2)\right) \exp\left(m + \frac{\sigma^2}{2}\right) dt.$$

Bringing the terms independent of t out of the integral and carrying out simplification on fractional terms, we obtain

$$\mathbb{E}[X] = \exp\left(m + \frac{\sigma^2}{2}\right) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}(t - \sigma t)^2\right) dt.$$

Here, it is easy to see that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}(t - \sigma t)^2\right) dt.$$

defines the integral of the pdf of a normally distributed random variable with a mean of σ and unit variance. Since this integral attains a value of 1, $\mathbb{E}[X]$ is reduced to

$$\mathbb{E}[X] = \exp\left(m + \frac{\sigma^2}{2}\right). \tag{12}$$

Thus,

$$\widehat{m} = \exp\left(m + \frac{\sigma^2}{2}\right). \tag{13}$$

Deriving variance of X

Now, we exploit the relation between variance and expected value of any random variable X, given by the following identity,

$$\mathbb{V}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2. \tag{14}$$

Since we have already computed $\mathbb{E}[X]$ in the previous step, this time we need to compute $\mathbb{E}[X^2]$. To that end, we need to take the following integral

$$\mathbb{E}[X^2] = \int_{\mathbb{R}} x^2 f(x) dx.$$

Arranging the limits and replacing $f(x)$ with Eq. 7, we obtain

$$\mathbb{E}[X^2] = \int_0^\infty x^2 \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\ln(x) - m)^2}{2\sigma^2}\right) \frac{1}{x} dx. \tag{15}$$

Simplifying the quadratic terms of x

$$\mathbb{E}[X^2] = \int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\ln(x) - m)^2}{2\sigma^2}\right) x dx. \tag{16}$$

We apply again the change of variables given in Eq.9. This time, we treat Eq.9 so as to achieve a term of xdx. From Eq. 9, we know that

$$\sigma x dt = dx.$$

Multiplying both sides with x

$$\sigma x^2 dt = x dx. \tag{17}$$

Writing x in terms of t using Eq. 9, we get

$$x = \exp\left(t\sigma + m\right).$$

Thus, the relation between the derivative terms given in Eq. 17 is

$$\sigma \exp\left(2t\sigma + 2m\right) dt = x dx.$$

Replacing this term in Eq. 16 and arranging the limits, we obtain

$$\mathbb{E}[X^2] = \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\ln(x) - m)^2}{2\sigma^2}\right) dt.$$

Replacing x with t in Eq. 15, employing the relation between the derivative terms given in Eq. 10, and arranging the limits, we get

$$\mathbb{E}[X^2] = \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi\sigma^2}} \sigma \exp\left(2t\sigma + 2m\right) \exp\left(\frac{-1}{2} t^2\right) dt.$$

Inside the integral, we multiply and divide by the term $\exp(2\sigma^2 + 2m)$, and obtain

$$\mathbb{E}[X^2] = \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-1}{2}\left(t^2 - 4\sigma t + 4\sigma^2\right)\right) \exp\left(2\sigma^2 + 2m\right) dt.$$

Bringing the terms independent of t out of the integral, we obtain

$$\mathbb{E}[X^2] = \exp\left(2\sigma^2 + 2m\right) \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^\infty \sigma \exp\left(\frac{-1}{2}\left(t^2 - 4\sigma t + 4\sigma^2\right)\right) dt.$$

Here, it is easy to see that

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^\infty \sigma \exp\left(\frac{-1}{2}\left(t^2 - 4\sigma t + 4\sigma^2\right)\right) dt.$$

defines the integral of the pdf of a normally distributed random variable with a mean of 2σ and unit variance. Since this integral attains a value of 1, $\mathbb{E}[X^2]$ is reduced to

$$\mathbb{E}[X^2] = \exp\big(2\sigma^2 + 2m\big).$$

On the other hand, the term $\mathbb{E}[X]^2$ in Eq. 14, can easily found by using Eq. 12.

$$\begin{aligned}\mathbb{E}[X]^2 &= \left(\exp\left(m + \frac{\sigma^2}{2}\right)\right)^2 \\ &= \exp\big(2m + \sigma^2\big)\end{aligned}$$

Finally, replacing the explicit expressions of $\mathbb{E}[X^2]$ and $\mathbb{E}[X]^2$ in Eq. 14, we get

$$\mathbb{V}[X] = \exp\big(2\sigma^2 + 2m\big) - \exp\big(2m + \sigma^2\big).$$

Since $\mathbb{V}[X] = \widehat{\sigma}^2$, we can write,

$$\widehat{\sigma} = \sqrt{\exp\left(2\sigma^2 + 2m\right) - \exp\left(2m + \sigma^2\right)}. \tag{18}$$

Deriving statistics of Y in terms of $\mathbb{E}[X]$ and $\mathbb{V}[X]$

Note that the above proof gives the expressions of mean and variance of log-normally distributed random variable X, i.e. $\widehat{m}$ and $\widehat{\sigma}^2$, in terms of the statistics of the normally distributed random variable Y, i.e. m and σ. In what follows, we will establish the opposite relations given in Eq. 2 of the manuscript. Namely we will express m and σ in terms of $\widehat{m}$ and $\widehat{\sigma}^2$.

Let us start from Eq. 18, and take its square

$$\widehat{\sigma}^2 = \exp\big(2\sigma^2 + 2m\big) - \exp\big(2m + \sigma^2\big). \tag{19}$$

By arranging the exponential terms and taking the logarithm of Eq. 19, we get

$$\ln\left(\widehat{\sigma}^2\right) = \ln\Big(\exp\big(\sigma^2 + 2m\big)\big[\exp\big(\sigma^2\big) + 1\big]\Big).$$

This is equivalent to

$$\ln\left(\widehat{\sigma}^2\right) = \sigma^2 + 2m + \ln\big[\exp(\sigma^2) - 1\big] \tag{20}$$

We also take the logarithm of Eq. 13 and write m in terms of $\widehat{m}$ and σ as follows

$$m = \ln(\widehat{m}) - \frac{\sigma^2}{2} \tag{21}$$

Here, we can easily see that

$$\sigma^2 + 2m = 2\ln(\widehat{m}) \tag{22}$$

Replacing the $\sigma^2 + 2m$ term on the right hand side of Eq. 20 with the one given by Eq. 22, we obtain

$$\ln\widehat{\sigma}^2 = 2\ln\left(\widehat{m}\right) + \ln\big[\exp(\sigma^2) - 1\big] \tag{23}$$

Collecting the terms relating the statistics of X (i.e. $\widehat{m}$ and $\widehat{\sigma}^2$) and the statistics of Y (i.e. m and σ) on different sides, we obtain

$$\ln\big[\exp(\sigma^2) - 1\big] = \ln\widehat{\sigma}^2 - 2\ln\left(\widehat{m}\right)$$

Using the properties of logarithm, this reduces to

$$\ln\big[\exp(\sigma^2) - 1\big] = \ln\left(\frac{\widehat{\sigma}^2}{\widehat{m}^2}\right)$$

Taking exponential of both sides and moving the constant term (1) to right hand side, we get

$$\exp(\sigma^2) = \frac{\widehat{\sigma}^2}{\widehat{m}^2} + 1$$

Taking the logarithm,

$$\sigma^2 = \ln\left(\frac{\widehat{\sigma}^2}{\widehat{m}^2} + 1\right) \tag{24}$$

Note that this completes the proof of Eq. 2.

Let us prove also Eq.1. For this purpose, let us replace σ^2 found in Eq. 24 in Eq. 21. This gives us

$$m = \ln(\widehat{m}) - \frac{1}{2}\ln\left(\frac{\widehat{\sigma}^2}{\widehat{m}^2} + 1\right)$$

Carrying the coefficient $(\frac{-1}{2})$ of the logarithmic term into the parenthesis as a square root, we get

$$m = \ln(\widehat{m}) - \ln\left(\sqrt{\frac{\widehat{\sigma}^2}{\widehat{m}^2} + 1}\right)$$

Using the properties of logarithm, we can organize this equation as

$$m = \ln\left(\frac{\widehat{m}}{\sqrt{\frac{\widehat{\sigma}^2}{\widehat{m}^2} + 1}}\right)$$

This completes the proof of Eq.1.